PRESSURE FORCES IN DISPERSE TWO-PHASE FLOW

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Abstract—A mathematical model for the description of pressure forces in averaged two-phase flow equations for disperse fluid–fluid and solid–fluid flows is obtained. The currently used engineering treatment is compared with this model and is found to be adequate when the pressure near the particles, bubbles or droplets is not very different from the volume-average pressure. In the case where large velocity differences exist between the phases or, in bubbly flow, when the bubbles undergo rapid growth or collapse, the engineering treatment is insufficient. The role of the pressure field of the continuous phase in the momentum equation of the disperse phase is also clarified. Finally, the model is extended to include viscous forces.

1. INTRODUCTION

The problem of the correct mathematical description of pressure forces in averaged two-phase flow models is a long-standing one with far reaching implications for both the physical realism and the mathematical structure of the models (see, e.g. Bouré 1979; Sha & Soo 1979). Frequently the equation sets employed in engineering calculations do not conform to rigorously derived averaged equations but, nevertheless, appear to yield satisfactory results in many applications. The reconciliation of this conflict between engineering practice and mathematical rigour has been one of the motivations of the present study. To this end an attempt is made to clarify the mathematical and physical nature of pressure forces in the averaged momentum equations for general (fluid–fluid or fluid–solid) disperse flows. We believe that the reasons for the success of many engineering formulations (Wallis 1969; Harlow & Amsden 1975; Liles & Mahaffy 1979; Spalding 1980; Smith 1980) clearly emerge from our results, as do the limitations of such models. Our analysis leads to the identification of a new pressure contribution for the continuous phase which arises from the difference between the volume-average pressure and the pressure at the interphase boundary. The physical origin and significance of this new contribution are explained in detail and the inadequacy of similar terms proposed in the past is demonstrated. We also show that the gradient of the pressure in the continuous phase is present in the momentum equation of the disperse phase as well, irrespective of its nature (fluid or solid). Finally, the results are extended to include viscous forces.

Another feature of the present study is a particularly simple implementation of volume averaging which enables us to derive the conservation equations in a compact and transparent way.

After completion of this work a recent paper by Rietema & van den Akker (1983) came to our attention in which equations identical to some of our results are derived. The method used by these Authors however appears to be rather *ad hoc* (in particular they do not make use of the averaging theorem [4]) and does not give a ready interpretation of certain interphase force terms.
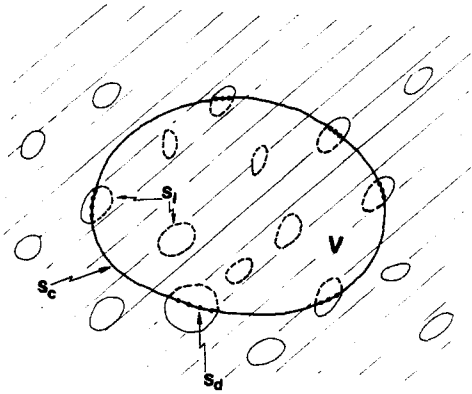
Figure 1. The averaging volume V with definitions of the outer bounding surface of the continuous phase, S_c, and of the disperse phase S_d. The interfaces between the phases within the averaging volume are denoted by S_i.

2. THE GENERAL VOLUME-AVERAGED CONVERSION EQUATIONS

The standard derivations of volume-averaged two–phase flow equations use the local instantaneous form of the equations as the starting point for the averaging procedure. However, these local equations are themselves the result of balances over finite volumes (see, e.g. Serrin 1959). Therefore it appears logically more satisfactory and, as it turns out, technically simpler, to use these volume balances as the starting point for the averaged equations.

In the region occupied by the two-phase mixture we take each point x to be surrounded by an averaging volume V. For simplicity we take V to be independent of both x and time, although allowance for these dependencies could be introduced into our procedure if so desired. The portion of V which is occupied by the phase k is denoted by V_k. In general, the boundary of V_k consists of a portion S_k of the boundary S of V, and of interfaces contained within V which we collectively denote by S_i. Figure 1 shows the surface of the averaging volume V broken up into its constituent parts S_c, the portion lying in the continuous phase, and S_d, the portion in the disperse phase. Note that $S_c + S_i$ is a closed surface which encloses the continuous phase present within V, and similarly, $S_d + S_i$ encloses the disperse phase.

Let now F_k be a general conserved quantity relative to phase k, distributed with density f_k per unit of mass. We can write the following balance equation for the total amount of F_k contained within the averaging volume:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{V_k} f_k \rho_k \, \mathrm{d}V_k = \int_{S_k} (-\rho_k f_k \mathbf{u}_k + \boldsymbol{\phi}_k) \cdot \mathbf{n} \, \mathrm{d}S_k$$

$$+ \int_{S_i} [-\rho_k f_k (\mathbf{u}_k - \mathbf{v}) + \boldsymbol{\phi}_k] \cdot \mathbf{n} \, \mathrm{d}S_i + \int_{V_k} \rho_k \theta_k \, \mathrm{d}V_k. \qquad [1]$$

In this equation $\boldsymbol{\phi}_k$ is the non-convective part of the flux with which F_k is transported, θ_k is the rate of production of F_k per unit of mass, and ρ_k and $\mathbf{u}_k$ denote the density and velocity fields of phase k. The first term in the r.h.s. of [1] accounts for the total transport of F_k into V_k across the boundary of the averaging volume which is stationary. The second term accounts for the transport across the interfaces contained within V, which are in general in motion with velocity $\mathbf{v}$. The unit normal $\mathbf{n}$ is defined as directed outward from the phase k.

In order to derive from [1] the averaged equation we introduce the volume fraction of
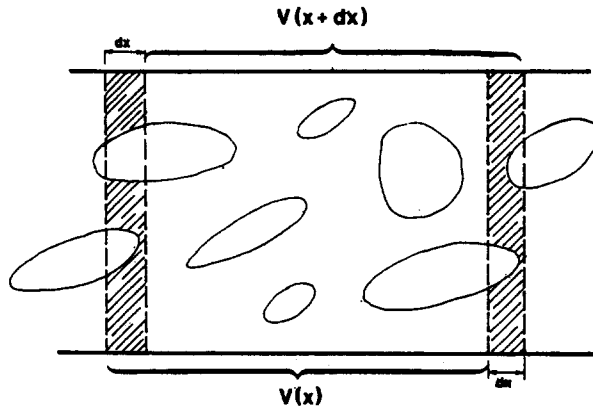
Figure 2. Illustration of the concept behind Slattery's theorem, [4], for a one-dimensional situation.

phase k

$$\alpha_k = V_k/V \qquad [2]$$

and the following definition of the (instantaneous) volume average for any scalar, vector, or tensor quantity q_k

$$\langle q_k \rangle = \frac{1}{V_k} \int_{V_k} q_k \, \mathrm{d}V_k. \qquad [3]$$

We also make use of the following theorem, which relates the surface integral of any vector or tensor-valued quantity ψ_k to the divergence of its volume average

$$\int_{S_k} \psi_k \cdot n \, \mathrm{d}S_k = V \cdot \int_{V_k} \psi_k \, \mathrm{d}V_k. \qquad [4]$$

Proofs of this important result can be found in Slattery (1967), Whitaker (1969) and Gray & Lee (1977). A very simple proof is also given in Prosperetti (1984).

The basic idea behind this result is illustrated with reference to figure 2. Consider a one-dimensional situation. Then the r.h.s. of [4] is essentially the difference of two volume integrals located at $x + \mathrm{d}x$ and x, divided by $\mathrm{d}x$, for $\mathrm{d}x \to 0$. The part common to the two integration domains cancels, and only the contributions of two "slices" of thickness $\mathrm{d}x$ survive, which are clearly the l.h.s. of [4].

With [2]–[4], making use of the constancy of V, we obtain from [1] the following general form of the volume-averaged conservation equation

$$\frac{\partial}{\partial t} (\alpha_k \langle \rho_k f_k \rangle) + V \cdot (\alpha_k \langle \rho_k f_k u_k \rangle) = V \cdot (\alpha_k \langle \phi_k \rangle) + \alpha_k \langle \rho_k \theta_k \rangle$$

$$+ \frac{1}{V} \int_{S_i} [-\rho_k f_k (u_k - v) + \phi_k] \cdot n \, \mathrm{d}S_i. \qquad [5]$$

For the case in which the conserved quantity F_k is momentum, we have $f_k = u_k$, $\theta_k = g$, the body force, and $\phi_k = -p_k I + \tau_k$ where p_k is the pressure, τ_k is the viscous stress tensor, and

$\underline{I}$ is the 3×3 identity tensor. In this case [5] becomes

$$\frac{\partial}{\partial t} (\alpha_k \langle \rho_k \mathbf{u}_k \rangle) + \nabla \cdot (\alpha_k \langle \rho_k \mathbf{u}_k \mathbf{u}_k \rangle) = -\nabla (\alpha_k \langle p_k \rangle) + \nabla \cdot (\alpha_k \langle \underline{\tau}_k \rangle) + \frac{1}{V} \int_{S_i} (-p_k \mathbf{n} + \underline{\tau}_k \cdot \mathbf{n}) \, \mathrm{d}S_i$$

$$-\frac{1}{V} \int_{S_i} \rho_k \mathbf{u}_k (\mathbf{u}_k - \mathbf{v}_i) \cdot \mathbf{n} \, \mathrm{d}S_i + \alpha_k \langle \rho_k \rangle \mathbf{g} . \qquad [6]$$

The pressure terms present in this equation are

$$\underline{\Pi}_k = -\nabla (\alpha_k \langle p_k \rangle) - \frac{1}{V} \int_{S_i} p_k \mathbf{n} \, \mathrm{d}S_i . \qquad [7]$$

We shall examine these terms in detail in the following.

3. THE PRESSURE TERMS

Expanding the gradient term in [7] we find

$$\underline{\Pi}_k = -\alpha_k \nabla \langle p_k \rangle - \langle p_k \rangle \nabla \alpha_k - \frac{1}{V} \int_{S_i} p_k \mathbf{n} \, \mathrm{d}S_i . \qquad [8]$$

An alternative form for $\underline{\Pi}_k$ can be obtained with the aid of an identity which is a consequence of [4]. If one takes $\underline{\psi}_k = \underline{I}$ in this equation, one finds

$$\nabla \alpha_k = \frac{1}{V} \int_{S_k} \mathbf{n} \, \mathrm{d}S_k , \qquad [9]$$

or, recalling that $S_k + S_i$ is a closed surface,

$$\nabla \alpha_k = -\frac{1}{V} \int_{S_i} \mathbf{n} \, \mathrm{d}S_i . \qquad [10]$$

from which

$$\underline{\Pi}_k = -\alpha_k \nabla \langle p_k \rangle - \frac{1}{V} \int_{S_i} (p_k - \langle p_k \rangle) \mathbf{n} \, \mathrm{d}S_i . \qquad [11]$$

The current engineering practice consists in modelling $\underline{\Pi}_k$ as:

$$\underline{\Pi}_k^E = -\alpha_k \nabla \langle p_k \rangle + \mathbf{F}_d + \mathbf{F}_{am} , \qquad [12]$$

where $\mathbf{F}_d$ and $\mathbf{F}_{am}$ indicate the drag and added-mass forces per unit volume. In what follows we shall investigate the validity of this approximation in the case of disperse two-phase flow. Comparing this engineering approximation with [11] we see that the two could coincide if $\mathbf{F}_d + \mathbf{F}_{am}$ could be identified with the integral term in [11]. As will be shown in the following sections, this identification is at best an approximation, which is not always valid. We may note that the second term in [8] has caused a long-standing controversy (see, e.g. Sha & Soo 1979; Bouré 1979) because, even in a situation of uniform pressure and vanishing velocity, it appears able to give rise to a flow if the spatial distribution of phase k is non-uniform. It is easy to show with the aid of [10] that no such flow could occur

since, if the pressure is uniform, $\langle p_k \rangle = p_k$, and [8] becomes

$$\Pi_k = -p_k \left(\nabla \alpha_k + \frac{1}{V} \int_{S_i} \mathbf{n} \, dS_i \right),$$

which vanishes by [10] as it should. Note that the integrals over the interfaces in [9] and [10] do not vanish because, for consistency of the averaging process, the surface of the averaging volume has to "cut through" the disperse phase (Nigmatulin 1979; Prosperetti 1984).

The difficulty with the term $\langle p_k \rangle \nabla \alpha_k$ arises only if the $\mathbf{F}_d + \mathbf{F}_{am}$ in [12] is identified with the last term in [8], which is the integral of the phase pressure over the interfaces, rather than with the second term in [11], which is the integral of the *excess* pressure over the interfaces.

In closing this section we wish to observe that, on the basis of [10], a surface average pressure $\langle\langle p_k \rangle\rangle$ is sometimes introduced by writing (Ishii 1975; Agee et al. 1980)

$$-\frac{1}{V} \int_{S_i} p_k \mathbf{n} \, dS_i = -\frac{1}{V} \langle\langle p_k \rangle\rangle \int_{S_i} \mathbf{n} \, dS_i = \langle\langle p_k \rangle\rangle \nabla \alpha_k .$$

Actually this definition is too restrictive since it implies that the surface integral vanishes whenever $\nabla \alpha_k = 0$, which is not necessarily true. An explicit expression for this surface integral will be given in section 5. It will be seen that the concept of a surface average pressure is indeed useful, but will have to be defined differently to avoid inconsistencies.

4. PRELIMINARIES

Although the conclusions to be arrived at here hold for general two-phase disperse flows, for definiteness we shall treat the case of bubbly flow, indicating along the way the changes required for other disperse flows. We shall use indices L and G to indicate the liquid and the gas, i.e. the continuous and the disperse phase, respectively. Clearly, by the definition of volume fractions,

$$\alpha_L + \alpha_G = 1 .$$

In a multiphase flow, as in any composite medium, one typically deals with "microscopic" fields exhibiting rapid spatial fluctuations over distances comparable with the size of the inhomogeneities. The purpose of averaging is to filter out these presumably unimportant fluctuations leaving relatively smooth, slowly-varying quantities which we may term macroscopic. In section 2 this filtering was partially achieved by the introduction of the averaged quantities defined by [3].

The general volume-averaged conservation equation [5], however, also contains the actual "microscopic" fields within the interface integrals. The principal aim of this paper is to express these terms by means of average quantities. Formal techniques based on the explicit identification of slow and rapid space scales have been developed (Sanchez-Palencia 1980; Papanicolau 1978) but so far they appear well suited only for linear problems and hence we proceed heuristically. We break up the pressure in the liquid into two parts

$$p = p_r + p_s ,$$ [14]

where the indices r and s denote the rapidly and slowly varying components, respectively, and the index L has been dropped for the time being. If a single bubble is moving in an

otherwise stagnant liquid [14] simply corresponds to breaking up the pressure into its static and dynamic parts, and the added mass and form drag forces on the bubble are given by

$$\mathbf{f}_p = -\oint_{S_b} p_r \mathbf{n}_b \, dS_b \qquad\qquad\qquad [15]$$

since p_r, being a constant in this case, makes no contribution. Here S_b is the bubble surface and $\mathbf{n}_b$ the outward-directed unit normal. If now the liquid is flowing, p_s is the pressure distribution causing the flow and [15] no longer represents the total force on the bubble, but only the part due to its relative motion in the liquid. The total force on the bubble is now

$$\mathbf{f} = \mathbf{f}_p - \oint_{S_b} p_s \mathbf{n}_b \, dS_b \, .$$

In view of the slow spatial variation of p_s, in performing the integration we may set $p_s \cong p_s(\mathbf{x}_b) + (\mathbf{x} - \mathbf{x}_b) \cdot \nabla p_s$, where $\mathbf{x}_b$ is a point in the bubble, to find

$$\mathbf{f} = \mathbf{f}_p - v_b \nabla p_s , \qquad\qquad\qquad [16]$$

where v_b is the bubble volume and the identity

$$\oint_{S_b} (\mathbf{x} - \mathbf{x}_0)_j n_{bk} \, dS_b = \delta_{jk} v_b , \qquad\qquad\qquad [17]$$

has been used. Essentially the same argument as leads to [16] can be found in Landau & Lifshitz (1959), section 11.

In a flowing liquid containing many bubbles we may apply a similar line of reasoning. Each bubble senses its environment through the pressure and viscous stresses acting on its surface. If the bubbles are not too close together on the average one may expect that the macroscopic liquid motion, non-uniformities in the bubble distribution, etc. will affect only the slowly varying component of the pressure (and the viscous stresses, which we neglect for the moment). To a first approximation, as far as p_s is concerned, the bubbly liquid is locally uniform with the average macroscopic properties. This remark is important because it furnishes a prescription for the evaluation of expressions such as [15] without the complications due to macroscopic non-uniformities in the mixture.

The preceding discussion suggests that the slowly varying component p_s be approximately identified with the volume-averaged pressure field $\langle p \rangle$ in the liquid, and within an averaging volume centered at $\mathbf{x}$ we represent p_s by:

$$p_s(\mathbf{x}') \cong \langle p \rangle(\mathbf{x}) + (\mathbf{x}' - \mathbf{x}) \cdot \nabla \langle p \rangle . \qquad\qquad\qquad [18]$$

Finally we define the surface average of the rapidly varying pressure on the liquid side of each bubble by

$$\bar{p} = \frac{1}{S_b} \oint_{S_b} p_r \, dS_b \qquad\qquad\qquad [19]$$

which allows us to write

$$p_r = \bar{p} + p_r' \qquad\qquad\qquad [20]$$

where p_r' is the deviation of p_r from its mean value over the surface of a bubble.

It is obvious that one gets the same result by computing $\mathbf{f}_p$ as the integral of p_r or of p_r'. $\mathbf{f}_p$ is determined by the variation of the pressure over the bubble surface, but not by the average level $\bar{p}$ of this pressure which, we note, can have an order of magnitude very different from that of p_r'.

5. PRESSURES FORCES ON THE LIQUID PHASE

We can now turn to the evaluation of the interface integral in [7]. In this integral we have contributions from bubbles completely contained in the averaging volume and from incomplete bubbles cut across by the surface of V. For one of the former we find by [14], [18], [17], [15],

$$\oint_{S_b} p\mathbf{n}\,dS_b = \oint_{S_b} [\langle p \rangle + (\mathbf{x} - \mathbf{x}') \cdot \nabla \langle p \rangle + p_r]\mathbf{n}\,dS_b = -v_b \nabla \langle p \rangle + \mathbf{f}_p. \qquad [21]$$

In this equation $\mathbf{n}$ is the unit normal directed away from the liquid and hence opposite to $\mathbf{n}_b$ used in the previous section. For convenience we temporarily drop the index L. The first term in [21] acts on the bubble like a buoyancy force in hydrostatics, directed towards regions of low average pressure. The explicit identification of this contribution to the total force on a bubble is an essential step in the derivation of the momentum equation of the disperse phase which will follow.

For an incomplete bubble (figure 3) we have identically

$$\int_{A_b} p\mathbf{n}_b\,dA_b = \int_{A_b} (p_s + \bar{p})\mathbf{n}_b\,dA_b + \int_{A_b} (p_r - \bar{p})\mathbf{n}_b\,dA_b.$$

Since, as remarked at the end of the previous section, the form drag and added mass force $\mathbf{f}_p$ is determined by the deviations of p_r from $\bar{p}$ over the bubble surface, the second term of this relation can be identified with a fraction of $\mathbf{f}_p$. This identification would be incorrect if the average surface pressure $\bar{p}$ had not been subtracted from p_r because A_b is an open surface. To evaluate the first term we add and subtract the contribution ΔS_G of that part of
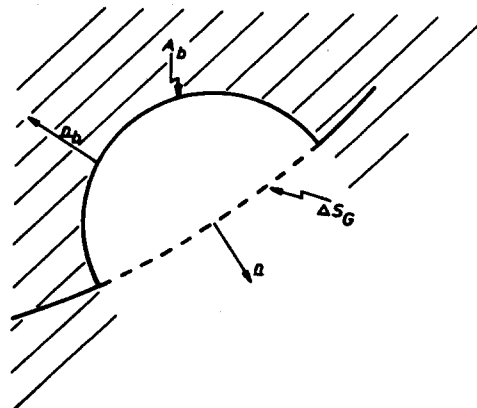

Figure 3. The contribution from the surface of an incomplete bubble. ΔS_G is the portion of the surface of the averaging volume lying within the bubble. $\mathbf{n}$ is the outwardly directed unit normal to the surface of the averaging volume and $\mathbf{n}_b$ is the normal on the interface of A_b directed outward from the liquid phase.

the surface of the averaging volume lying within the bubble (see figure 3):

$$\int_{A_b} (p_s + \bar{p})\mathbf{n}_b \, dA_b = \int_{A_b} p_s\mathbf{n}_b \, dA_b + \int_{\Delta S_G} p_s\mathbf{n} \, dA_b - \int_{\Delta S_G} (p_s + \bar{p})\mathbf{n} \, dS_G,$$ [22]

where we have made use of the fact that $\bar{p}$ is a constant so that the integral over A_b is minus that over ΔS_G. The first two terms are the integral of p_s over the closed surface A_b plus ΔS_G and, in view of [18], can be evaluated by using [17] to find $-v_b'\nabla\langle p\rangle$, where v_b' is the part of v_b lying inside the averaging volume. Combining the contributions of all the complete and incomplete bubbles in V we then find

$$\frac{1}{V}\int_{S_i} p\mathbf{n} \, dS_i = -\alpha_G\nabla\langle p\rangle + \mathbf{F}_p + \frac{1}{V}\int_{S_G} (p_s + \bar{p})\mathbf{n} \, dS_G,$$ [23]

where we have used $\Sigma v_b = \alpha_G V$ and

$$\mathbf{F}_p = \frac{1}{V}\int_{S_i} (p_r - \bar{p})\mathbf{n} \, dS_i,$$ [24]

is the total added mass and drag force per unit volume on the bubbles. This quantity will be of the order of the product of the bubble number density and $\mathbf{f}_p$. For the last term of [23] we note that from [18]

$$\int_{S_G} p_s\mathbf{n} \, dS_G = \langle p\rangle\int_{S_G} \mathbf{n} \, dS_G + \nabla\langle p\rangle \cdot \int_{S_G} (\mathbf{x}' - \mathbf{x})\mathbf{n} \, dS_G = V(\langle p\rangle\nabla\alpha_G + \alpha_G\nabla\langle p\rangle)$$ [25]

where the first term follows from [9] and the second one is computed in appendix A. Furthermore, by [4],

$$\frac{1}{V}\int_{S_G} \bar{p}_L\mathbf{n} \, dS_G = \nabla(\alpha_G\langle\bar{p}_L\rangle^G).$$ [26]

In the l.h.s. we have the integral of the liquid quantity $\bar{p}_L$ over surfaces within the gas phase. However, since $\bar{p}_L$ has a unique value for each bubble, this integral is well defined. The average in the r.h.s. is over the gas phase as is indicated by the superscript G. A physical interpretation of this term will be given in the next section.

Inserting [25] and [26] into [23] we finally obtain

$$\frac{1}{V}\int_{S_i} p_L\mathbf{n} \, dS_i = -\alpha_G\nabla\langle p_L\rangle + \mathbf{F}_p + \nabla[\alpha_G(\langle p_L\rangle + \langle\bar{p}_L\rangle^G)],$$ [27]

from which the total liquid pressure force per unit volume, [7], is

$$\Pi_L = -\nabla(\alpha_L\langle p_L\rangle) + \alpha_G\nabla\langle p_L\rangle - \mathbf{F}_p - \nabla[\alpha_G(\langle p_L\rangle - \langle\bar{p}_L\rangle^G)],$$

or, using $\alpha_L = 1 - \alpha_G$, in the first term,

$$\Pi_L = -\alpha_L\nabla\langle p_L\rangle - \mathbf{F}_p - \nabla(\alpha_G\langle\bar{p}_L\rangle^G).$$ [28]

We may remark that the term $\alpha_G\nabla\langle p_L\rangle$ which reflects the tendency of the bubbles to move

towards regions of low macroscopic pressure is essential in obtaining $\alpha_L \nabla \langle p_L \rangle$ from $\nabla(\alpha_L \langle p_L \rangle)$.

We can also compare our result [27] with the definition of surface pressure given at the end of section 3. Expanding the gradient term in [27] and using $\nabla \alpha_G = -\nabla \alpha_L$ we find

$$-\frac{1}{V} \int_{S_i} p_L \mathbf{n} \, dS_i = (\langle p_L \rangle + \langle \bar{p}_L \rangle^G) \nabla \alpha_L - \mathbf{F}_p - \alpha_G \nabla \langle \bar{p}_L \rangle^G \, .$$

The term $\langle p_L \rangle + \langle \bar{p}_L \rangle^G$ can be identified with a surface average pressure. A term corresponding to $\mathbf{F}_p$ is usually accounted for separately in treatments which introduce $\langle\!\langle p_L \rangle\!\rangle$. The last term is however new and cannot be written in the form of a surface pressure multiplied by $\nabla \alpha_L$. The presence of this last term removes the inconsistency in the definition of $\langle\!\langle p_L \rangle\!\rangle$ given at the end of section 3 since it will give a contribution when $\nabla \alpha_L = 0$, provided that $\nabla(\langle \bar{p}_L \rangle^G)$ is non zero.

6. DISCUSSION

Equation [28] represents the resultant of the pressure forces acting on the continuous phase, which we have taken to be the liquid for definiteness although an identical result would be found if the continuous phase were the gas. This result admits of a simple physical interpretation in the following terms.

Consider the slowly varying pressure component first. With reference to figure 4, if the pressure everywhere in the shaded region bounded by the heavy line were $\langle p_L \rangle$, the pressure force on the material in the shaded region would be $-(V_L + V_{G,c}) \nabla \langle p_L \rangle$, when $V_{G,c}$ is the total volume of the bubbles completely contained in V. To obtain the force on the liquid only we must subtract the pressure force exerted by the internal bubbles, which is the buoyancy force identified in [21], amounting to $-V_{G,c} \nabla \langle p_L \rangle$. The result is then $-V_L \nabla \langle p_L \rangle$, or $-\alpha_L \nabla \langle p_L \rangle$ per unit volume. We must now correct this for the fact that the pressure acting on the liquid at the interfaces S_i is in fact not $\langle p_L \rangle$ but $\langle p_L \rangle + p'_L + \bar{p}_L$ (see [14] and [20]). The second term, upon integration over a bubble surface, gives rise to the added mass and pressure drag forces on that bubble. The sum of all such contributions is $\mathbf{F}$, the sum of the added mass and pressure drag forces per unit volume. For the last term $\bar{p}_L$, which is constant for each bubble, only the incomplete bubbles make any contribution. Referring to figure 3, the integration can be carried out over ΔS_G rather than A_b, and by the averaging theorem [3] the contribution of all the incomplete bubbles is $\nabla(\alpha_G \langle \bar{p}_L \rangle^G)$, defined by [26].
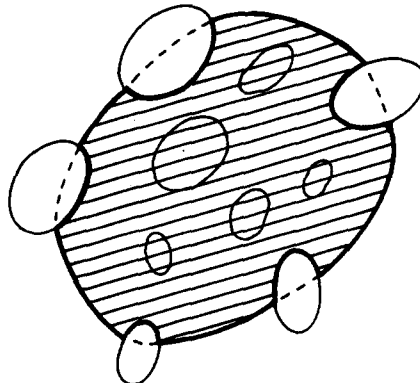

Figure 4. Interpretation of the result [28].

Upon comparison of the result [28] with the usual engineering approximation [12], it is seen that the two coincide apart from the last term. As can be seen from [19], this term is small when the average pressure over the surface of the bubbles is close to the volume-average pressure $\langle p_L \rangle$. One case in which this approximation does not hold is that of large relative velocity between the phases. As a simple example, from the potential flow pressure distribution around an isolated sphere (Landau & Lifshitz 1959) and the definition [19] we obtain

$$\bar{p}_L = -\frac{1}{4}\rho_L(\mathbf{u}_G - \mathbf{u}_L)^2$$

so that

$$\langle \bar{p}_L \rangle^G = \frac{1}{V_G}\int_{V_G} \bar{p}_L \, dV_G \cong -\frac{1}{4}\rho_L(\mathbf{u}_G - \mathbf{u}_L)^2 \,,$$

if α_G is small and $\mathbf{u}_G - \mathbf{u}_L$ is approximately constant over V_G.

Another example in which $\bar{p}_L$ may be significant is that of a growing or collapsing bubble since, near the bubble, a pressure gradient must be present in the liquid to cause a motion in the radial direction. In the case of an isolated bubble of radius R at rest relative to the liquid, the difference between the average surface pressure and the pressure "at infinity", which may be identified with $\langle p_L \rangle$ (van Wijngaarden 1968), is given by the Rayleigh–Plesset equation (Plesset & Prosperetti 1977):

$$\bar{p}_L = \rho_L\left[R\frac{d^2R}{dt^2} + \frac{3}{2}\left(\frac{dR}{dt}\right)^2\right].$$

The term $V(\alpha_G\langle \bar{p}_L \rangle^G)$ acts as a source of momentum for the liquid, the physical significance of which, in this second example, can be appreciated by considering the situation shown in figure 5, which refers to the case of bubbly flow in a pipe with a spatial variation of the number of bubbles. If the bubbles are expanding, $\langle \bar{p}_L \rangle^G > 0$ and the term in question tends to increase the velocity of the liquid in the direction in which α_G decreases. The bubbles tend to "push away" the liquid and part of the radial momentum associated with their expansion is converted into linear momentum along the direction of the flow. Similarly, if α_G is uniform but $\langle \bar{p}_L \rangle^G$ is not because some bubbles are expanding more rapidly than others, the liquid tends to flow away from the region occupied by the more rapidly expanding bubbles.

7. PRESSURE FORCES ON THE DISPERSE PHASE

Let us now turn to the pressure forces acting on the disperse phase. If viscous forces and change of phase effects are neglected the pressure in the gas, which we shall denote
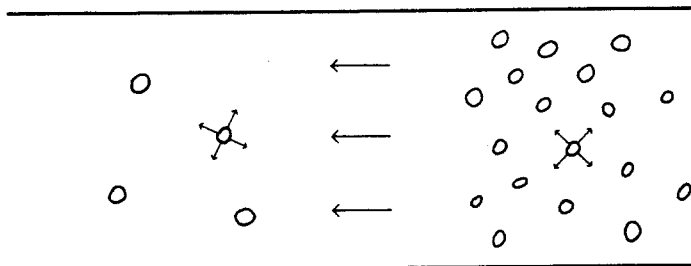

Figure 5. Sketch of the flow induced in a pipe by a non-uniform population of expanding bubbles.

by p_G, is related to the pressure p_L in the liquid phase by

$$p_G - p_L = \zeta C,$$ [29]

where ζ is the surface tension coefficient and C is the interfacial curvature. The contribution coming from a typical *internal* bubble is then

$$\oint_{S_b} p_G \mathbf{n}_b \, dS_b = \oint_{S_b} p_L \mathbf{n}_b \, dS_b + \oint_{S_b} \zeta C \mathbf{n}_b \, dS_b.$$ [30]

Observing that the normal appearing here is the opposite of the one used for the continuous phase, it is seen that the first integral in the r.h.s. is the negative of the one appearing in the l.h.s. of [21]. For constant surface tension, the second integral vanishes since a force internal to the bubble surface cannot give rise to a net resultant. A formal proof of this fact is contained in appendix B.

As for each bubble *cut across* by the surface of the averaging volume, we consider together the contribution of the interface and of the portion of the surface of V within the bubble. With reference to the notation of figure 3 we then have the following expression for the total pressure force on the gas of an incomplete bubble:

$$\mathbf{f}_b = -\int_{A_b} p_G \mathbf{n}_b \, dA_b - \int_{\Delta S_G} p_G \mathbf{n} \, dS_G,$$ [31]

which, since p_G is very nearly uniform in the bubble, can be evaluated approximately using the Taylor series expansion $p_G \cong p_{G0} + (\mathbf{x} - \mathbf{x}_0) \cdot \boldsymbol{V} p_G$ and [17] to obtain

$$\mathbf{f}_b = -v'_b \boldsymbol{V} p_G,$$ [32]

where v'_b is the part of the volume v_b of the bubble contained within the averaging volume. On the other hand, to the same approximation, the integral in the l.h.s. of [30] is $v_b \boldsymbol{V} p_G$ so that, from [30] and [21],

$$v_b \boldsymbol{V} p_G \cong + \int_{S_b} p_L \mathbf{n}_b \, dS_b = v_b \boldsymbol{V} \langle p_L \rangle - \mathbf{f}_p$$

and, substituting in [32], we find

$$\mathbf{f}_b = -v'_b \boldsymbol{V} \langle p_L \rangle + \frac{v'_b}{v_b} \mathbf{f}_p.$$ [33]

In conclusion, adding together all the contributions from the complete and the incomplete bubbles we find

$$\boldsymbol{\Pi}_G = \mathbf{F}_p - \alpha_G \boldsymbol{V} \langle p_L \rangle.$$ [34]

This result also applies when the disperse phase is a solid. In this case for the internal particles [30] holds with $\zeta = 0$ and $p_G \mathbf{n}$ replaced by $\underline{\tau}_i \cdot \mathbf{n}$, where $\underline{\tau}_i$ is the stress in the solid at the interface. The r.h.s. of [32] would be replaced by $v'_b \nabla \cdot \underline{\tau}_i$, again leading to [33]. In the case of a liquid disperse phase the same relations hold, provided that the pressure field within each droplet can be approximated by a linear function of position.

The result [34] is precisely of the engineering type [12]. The same pressure gradient is seen to act on the disperse phase as on the continuous phase. The physical reason lies in the fact that the internal pressure in each particle has no relevance for the translational motion of that particle, which is caused rather by the pressure distribution in the surrounding medium. Of course, at high particle concentrations collisions may also contribute a stress-like term to the disperse phase momentum equation (see, e.g. Savage 1983). In our formulation this situation would manifest itself as a growth in importance of the rapidly varying pressure contribution and would appear in the force per unit volume F.

In conclusion, adding the results for the pressure forces on the continuous and disperse phases given by [28] and [34] we obtain the following expression for the total pressure force per unit volume of the mixture:

$$\Pi_L + \Pi_G = -\nabla\left[\langle p_L \rangle + \alpha_G \langle \bar{p}_L \rangle^G\right],$$

or, equivalently

$$\Pi_L + \Pi_G = -\nabla\left[\alpha_L \langle p_L \rangle + \alpha_G(\langle p_L \rangle + \langle \bar{p}_L \rangle^G)\right]. \tag{35}$$

It is interesting to consider the meaning of this result in the case of bubbly flow in a pipe. The net pressure force on the mixture contained between the sections L_1 and L_2 of figure 6 is

$$-\int_{S_L} p_L \mathbf{n}\, dS_L - \int_{S_G} p_G \mathbf{n}\, dS_G = -V\nabla(\alpha_L\langle p_L \rangle + \alpha_G\langle p_G \rangle), \tag{36}$$

where use has been made of the averaging theorem [4] and S_G and S_l are the areas of the sections L_1 and L_2 occupied by the gas and liquid phases, respectively. This result differs from [35] because $p_G \neq p_L + \bar{p}_L$ due to the surface tension force. However [36] is not the total force acting on the mixture through the planes L_1 and L_2. Surface tension forces also exist at the intersections of L_1 and L_2 with the bubble boundaries as shown by the arrows in figure 6. The effect of these forces is precisely to cancel the contribution to p_G coming from surface tension and therefore, when all the forces are correctly accounted for, [35] is obtained.
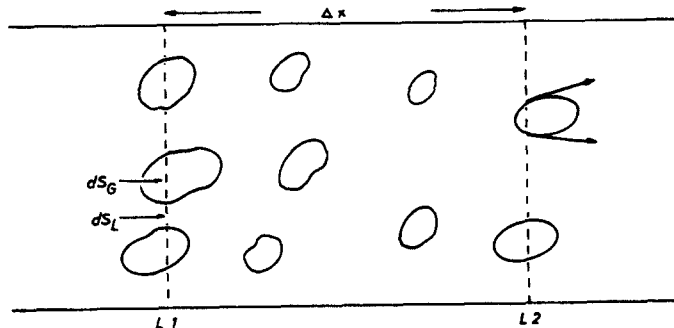

Figure 6. Sketch of bubbly flow in a pipe for the discussion of [35].

8. VISCOUS STRESSES

The previous considerations can be extended to take into account viscous forces on the continuous phase. Let us denote by $\mathbf{T}_k$ the resultant of the viscous stresses on the phase k; from [6] we have

$$T_k = V \cdot (\alpha_k \langle \underline{\tau}_k \rangle) + \frac{1}{V} \int_{S_i} \underline{\tau}_k \cdot \mathbf{n} \, dS_i \,.$$

Again, as in [14], we can split the viscous stress tensor into slowly and rapidly varying parts, $\underline{\tau} = \underline{\tau}_r + \underline{\tau}_s$, and the viscous drag force on each bubble is

$$\mathbf{f}_d = \oint_{S_b} \underline{\tau}_r \cdot \mathbf{n}_b \, dS_b \,. \tag{37}$$

The argument leading to [28] can be followed without modifications for each tensorial component τ_{ij} and the final result is

$$\mathbf{T}_L = \alpha_L V \cdot \langle \underline{\tau}_L \rangle - \mathbf{F}_d + V \cdot (\alpha_G \langle \bar{\underline{\tau}}_L \rangle^G) \,, \tag{38}$$

where $\mathbf{F}_d$ is given by an expression similar to [24] and is the total viscous drag force per unit volume. The surface average $\bar{\underline{\tau}}_L$ of the rapidly varying part depends on the relative motion between bubble and liquid. For instance, for Stokes flow around a solid sphere (which is appropriate for bubbles in water due to surface contamination), only the component of $\bar{\underline{\tau}}_L$ with both indices in the direction of the relative motion is non-zero and has the value

$$\bar{\tau}_{L11} = -\frac{1}{4} \frac{\mu}{R} (u_G - u_L) \,.$$

Combining [28] and [38] we have for the total stress forces on the liquid phase

$$\boldsymbol{\Pi}_L + \mathbf{T}_L = \alpha_L V \cdot \langle \underline{\sigma}_L \rangle - \mathbf{F} + V \cdot (\alpha_G \langle \bar{\underline{\sigma}}_L \rangle^G) \,, \tag{39}$$

where $\sigma_{ij} = -p\delta_{ij} + \tau_{ij}$ is the total stress tensor and $\mathbf{F} = \mathbf{F}_d + \mathbf{F}_p$. This result can of course also be obtained directly by following the same procedure leading to [28] using σ_{ij} in place of p. Similarly we may conclude that for the disperse phase

$$\boldsymbol{\Pi}_G + \mathbf{T}_G = \alpha_G V \cdot \langle \underline{\sigma}_G \rangle + \mathbf{F} \,. \tag{40}$$

9. CONCLUSIONS

In this paper we have analysed the pressure forces at work in a disperse two-phase flow. The key to the analysis has proven to be the recognition of the existence of two different components of the pressure in the continuous phase with slow and rapid spatial variation, respectively. The first component has been identified with the volume average pressure, and acts on each bubble, droplet or particle as an effective "buoyancy" force. The second component gives rise to the pressure drag and added mass forces. The combination of these effects is in agreement with recently used engineering models of two-phase flow. However, the rapidly varying spatial component is also responsible for an additional term [26] which has not previously been identified. Situations in which this new term can be important are those in which the pressure averaged over the surface of the bubble or particle is significantly different from the volume average pressure in the continuous phase. This

could happen at high relative velocity, or for expanding or collapsing bubbles. This latter situation is depicted in figure 5, where the need is evident for a force driving the liquid from right to left in response to the growth of the bubbles. The new term is present in the equation of motion for the continuous phase [28] but not in that for the disperse phase [34] since an average surface pressure cannot have any effect on the translational motion of an individual bubble, droplet or particle. The results are extended to include viscous forces in [39] and [40].

As a final point of interest our development includes in section 2 a very simple implementation of volume averaging, which exploits a theorem due to Slattery (1967), [4].

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APPENDIX A

Here we wish to obtain an approximate value for the integral

$$I_{ij} = \frac{1}{V} \int_{S_G} (\mathbf{x}' - \mathbf{x})_i \cdot n_j \cdot dS_G, \qquad [A1]$$

where $\mathbf{x}$ is the center of the averaging volume V and $\mathbf{x}'$ the variable of integration. From the point of view of an average description of the phases the value of this integral can only depend on the distribution of the gas as described by α_G and its derivatives and hence must have the form

$$I_{ij} = f(\alpha_G, |\nabla\alpha_G|, \nabla^2\alpha_G \ldots)\delta_{ij} + g(\alpha_G, |\nabla\alpha_G|, \nabla^2\alpha_G \ldots)\partial_i\alpha_G\partial_j\alpha_G. \qquad [A2]$$

(This presupposes that the averaging volume does not have any preferred direction, as is necessary for a consistent averaging procedure in three dimensions. A corresponding result however can also be derived for instance for the case of one-dimensional modelling of pipe flow.)

On purely dimensional grounds all the derivatives appearing in [A2] have to be multiplied by a length l related to the size of the averaging volume. On the other hand the derivatives themselves must be of order $1/L$, where L is a characteristic length of the system, and for averaging to be meaningful, one must have $l \ll L$. Therefore we can approximate [A2] by

$$I_{ij} \cong f(\alpha_G, 0, 0 \ldots)\delta_{ij} + O(l/L).$$

The dependence of f on α_G can be established by considering the case in which the bubbles are distributed uniformly over the surface of the averaging volume. It is obvious that in this situation doubling, for example, the number of bubbles, will also double the value of I_{ij}. Hence $f(\alpha_G, 0, 0, \ldots)$ must be proportional to α_G. To establish the value of the constant of proportionality we can consider the particular case $\alpha_G = 1$. The surface S_G is now closed and the integral in [A1] therefore has the value V. This shows that $f(\alpha_G, 0, 0, \ldots) = \alpha_G$ so that

$$I_{ij} = \alpha_G\delta_{ij} + O(l/L),$$

which is the result used in [25].

APPENDIX B

The net surface tension force over a closed surface

Consider a portion ΔS of a closed interface separating two fluids. The total surface tension force ΔF acting on ΔS can be expressed as the integral (Batchelor 1968)

$$\Delta F = -\oint_{\mathcal{L}} \varsigma \mathbf{n} \times d\mathbf{l},$$

where £ is the line bounding ΔS, n is the outward unit normal to ΔS, and the orientation of the line is related to that of the normal by the usual right-hand convention.

Consider now the projection of ΔF on an arbitrary constant vector e,

$$\Delta F \cdot e = -\oint_{£} \zeta e \cdot (n \times dl) = -\oint_{£} \zeta (e \times n) \cdot dl$$

or, by Stokes' theorem,

$$\Delta F \cdot e = -\int_{\Delta S} \nabla \times (\zeta e \times n) \cdot n \, dS.$$

If we consider the resultant surface tension force F on the whole surface S, we then have

$$F \cdot e = -\int_{S} \nabla \times (\zeta e \times n) \cdot n \, dS = 0, \qquad [B1]$$

by the divergence theorem.

However,

$$\nabla \times (\zeta e \times n) = e(n \cdot \nabla \zeta + \zeta \nabla \cdot n) - (e \cdot \nabla \zeta)n - \zeta(e \cdot \nabla)n,$$

and the first term vanishes since $\nabla \zeta$ is tangent to the surface, while the last term vanishes when dotted into n, since $[(e \cdot \nabla)n] \cdot n = e \cdot \nabla[(1/2)n^2] = 0$. Hence, by the vanishing of [B1] and the arbitrariness of e, we conclude that

$$\int_{S} \zeta(\nabla \cdot n)n \, dS = \int_{S} \nabla \zeta \, dS$$

which shows that the first integral vanishes for constant surface tension ζ. Since the local curvature C on S equals $\nabla \cdot n$, this proves the assertion made in the text.